Socially emergent cognition: particular outcome of student-to-student discursive interactions during mathematical problem solving

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Abstract
This article extends the socio-cultural notion about the cognitive consequences of discourse. Based on an empirical analysis of data of high school students engaged in problem solving, the study reported here posits the notion of socially emergent cognition as a process through which ideas and ways of reasoning materialize from the discursive interactions of interlocutors that go beyond those already internalized by any individual interlocutor. The context of the students' work is a combinatorial problem set in a non-Euclidean space, called Taxicab. Emerging from their discourse, the students articulate isomorphisms to solve a generalized version of the given problem.
Keywords: Discourse; Socially emergent cognition; Isomorphism.

Cognição socialmente emergente: resultados particulares de interações discursivas aluno–aluno durante a resolução de problemas matemáticos

Resumo
Este artigo aprofunda a noção sociocultural acerca das conseqüências cognitivas do discurso. Com base em uma análise empírica de dados sobre o modo como alunos de colegial solucionam um problema, o estudo aqui relatado propõe o conceito de cognição socialmente emergente como um processo mediante o qual idéias e modos de raciocínio afloram da interação discursiva de interlocutores, indo além daquelas já internalizadas por todo e qualquer indivíduo. O contexto do trabalho dos alunos é um problema combinatório situado num espaço não-euclidiano chamado Táxi. Emergindo de seu discurso, os alunos articulam isomorfismos para solucionar uma versão geral do problema dado.
Palavras-chave: Discurso; Cognição socialmente emergente; Isomorfismo.

In mathematics education, the topic of discourse and its cognitive influence in mathematics classrooms has been the subject of numerous studies. The importance of individual communication and group discussion in learning are processes about which mathematics educators agree. As Cobb, Boufi, McClain, and Whitenack (1997) note, consensus on this point within the mathematics education community transcends theoretical differences and include researchers who draw primarily on mathematics as a discipline, on constructivist theory, and on sociocultural perspectives.

This consensus notwithstanding, two broad lines of research can be distinguished. On the one hand, the nature of communications and how teachers can encourage and support communicative acts among students that are both productive and mathematical have been the subject of empirical research and theoretical reflection (for example, see Alrø; Skovsmose, 1998; Cobb; Boufi; McClain; Whitenack, 1997; Fernandez, 1994; Maher, 1998; O'Connor; Michaels, 1996; Seeger, 2002; Sfard, 2000, 2001; Sfard; Nesher; Streefland; Cobb; Mason, 1998). In these studies, teachers are seen as instrumental in triggering either students' reflective discourse or their otherwise productive discussions. As Cobb, Boufi, McClain, and Whitenack hypothesize (1997), learners' participation in reflective discourse "*constitutes conditions for the possibility of mathematical learning*" (p. 264, original emphasis). On the other hand, a question that arises is how and to what extent can such discussions occur among students themselves, particularly when teachers play a minimal role to none at all in triggering reflective discourse or otherwise mathematically productive discussions. This question points to others: Do these discussions enable students to go beyond exchanging information, providing a discursive context for their collective to achieve novel cognitive results by pushing against the boundaries of each others "zone of proximal development"? In what ways can interlocutors develop discursively mathematical ideas and reasoning that go beyond those of any individual interlocutor but that are later reflective of individual interlocutor's understanding? That is, is socially emergent cognition possible in settings of student-to-student discursive interactions?

Theoretical perspective

In this study, key terms include discourse, student-to-student or peer mathematical discussion, and

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socially emergent cognition. Discourse here refers to language (natural or symbolic, oral or gestic) used to carry out tasks – for example, social or intellectual – of a community. In agreement with Pirie and Schwarzenberger (1988), student-to-student or peer conversations are mathematical discussions when they possess the following four features: are purposeful, focus on a mathematical topic, involve genuine student contributions, and are interactive. Additionally, in the context of the data for this article, these student-to-student discursive collaborations involve minimal, substantive interaction with a teacher or researcher.

This study is informed by and goes beyond an essential tenet of Vygotskyan socio-cultural psychology of learning. A central tenet in Vygotsky's developmental theory is the notion of the "zone of proximal development" (Vygotsky, 1978). A child or learner's "zone of proximal development is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). The zone of proximal development includes all the functions and activities that a learner can perform only with the assistance of someone else. The "someone else" in a scaffolding process – rendering gradually diminishing support in response to learners' increased display of successful performance – could be an adult such as a teacher or another peer who has already mastered the particular function. Eventually, as Vygotsky states, "newly awakened processes…are internalized, they become part of the child's independent developmental achievement" (1978, p. 90).

Theoretically, the results of this investigation go further than the Vygotskyan mechanism by which learners individually transcend the boundaries of their zone of proximal development. Based on evidence from the data, this study posits the notion of socially emergent cognition as a process through which ideas and ways of reasoning materialize from the discursive interactions of interlocutors that go beyond those already internalized by any individual interlocutor. Working cognitively within and at the frontier their zone of proximal development, the interlocutors collaborate on a challenging mathematical task that not one interlocutor has already mastered and therefore scaffolds the thinking of the others. Instead, as a by-product of their engaged conversational interaction, evidenced in their interactional discourse, their mutually constituted ideas and ways of reasoning are subsequently internalized and later reflective of individual interlocutor's understanding. The product of their socially emergent cognition is not attributable to any one interlocutor but rather is a negotiated entity, constituted through discursive interactions, and eventually a shared part of the awareness of each interlocutor. This notion surfaced from analysis of features and functions of conversational exchanges among four students engaged collaboratively, without assigned roles, to understand and resolve an open-ended, combinatorial problem embedded in a non-Euclidean plane. The problem is presented in the next section.

Before discussing the task and the context out which the analysis arose, it is important to note literature that corroborates the notion of socially emergent cognition. After a grounded-theory analysis of the data from which the idea of socially emergent cognition materialized, I have become acquainted with the work of other researchers who have noted the potential and profit for learning that can occur from the collaboration among peers (for example, Damon; Phelps, 1991; Hutchins, 1995; Stahl, 2005). From a "social interactional" approach, Damon and Phelps (1991) examine how cognitive growth is stimulated among fourth-graders, collaborating in "dyads" assigned to work on either mathematics or spatial problems and the following year on balance-scale problems. They note features and outcomes of peer collaboration as a learning context. In an ethnographic study of ship navigation as a case of "naturally occurring culturally constituted human activity," Hutchins (1995) elaborates a notion of socially distributed cognition as computational processes resulting from practitioners engaged in a cultural activity. In one instance, he analyzes how a trained navigational team successfully resolved a potentially disastrous, novel situation that they encountered on the high seas. His analysis of this instance leads him to observe that "although the participants may have represented and thus learned the solution after it came into being, the solution was clearly discovered by the organization before it was discovered by any of the participants" (Hutchins, 1995, p. 351). From the perspective of computer-supported collaborative learning, Stahl (2005) presents a theory of group cognition as knowledge building at the level of small groups of students functioning within a computer environment. He calls for further empirical research "to clarify the nature of shared knowledge and group cognition" (p. 87).

The present study differs from the three investigations described above. This study analyzes the socially emergent cognition that results from the conversational interaction of four novice and nonhierarchically identified individuals who collaborate face-to-face to solve an open-ended but well-defined mathematical problem. Similar to each of the three investigations, this study bases its analysis on the discourse of its participants.

The analysis of conversational exchanges is informed by the work of Davis (1997), who inquires into teacher listening and its consequent impact on the growth of student understanding. This study builds on his inquiry. It also applies and extends Davis's categories to analyze discursive interactions of students engaged in discourse. This theoretical construct contains has four categories – evaluative, informative, interpretative, and negotiatory – described below, and guides the inquiry into how learners' discursive exchanges contribute to the mathematical ideas and reasoning that they evidence.

Evaluative: an interlocutor maintains a non-participatory and an evaluative stance, judging statements of his or her conversational partner as either right or wrong, good or bad, useful or not.

Informative: an interlocutor requests or announces factual data to satisfy a doubt, a question, or a curiosity (without evidence of judgment).

Interpretive: an interlocutor endeavors to tease out what his or her conversational partner is thinking, wanting to say, expressing, and meaning; an interlocutor engages an interlocutor to think aloud as if to discover his or her own thinking.

Negotiatory: an interlocutor engages and negotiates with his or her conversational partner; the interlocutors are involved in a shared project; each participates in the formation and the transformation of experience through ongoing questioning of the state of affairs that frames their perception and actions.

These are not mutually exclusive categories; a unit of meaningful conversation may have more than one interlocutory feature. Based on the theoretical perspective of this study and analysis of the data, socially emergent cognition arises from negotiatory interlocution in a collaborative problem-solving setting. It presupposes that interlocutors are engaged in a student-to-student, mathematical discussion with minimal teacher intervention.

Research context

This research lies within a longitudinal investigation, now in its 18th year, which traces the mathematical development of students while they solve open-ended but well-defined mathematical problems (cf., Maher, 2005).[1] As Weber, Maher, and Powell (in press) note, many of these problems are challenging in the sense that students often initially do not possess the procedural or algorithmic tools to solve the problems, but are asked to develop their own tools in the problem-solving context. In this environment, collaboration and justification are encouraged, and teachers and researchers do not provide explicit guidance on how problems should be solved. One aspect of the longitudinal study is that students work on *strands* of challenging tasks – or sequences of tasks that may differ superficially but pertain to a class of mathematical ideas. The use of strands of related, challenging tasks allows researchers to trace the development of students' reasoning about a particular class of mathematical ideas over long periods of time (e. g., Maher; Martino, 2000).

The participants are four students in their last year of high school, who are studying advanced high school mathematics and who, from their entry into first grade, have participated in mathematical activities of a longitudinal study on the development of mathematical ideas. For these twelve years of their participation in the longitudinal study, these students have engaged tasks from several strands of mathematics, including counting and combinatorics, algebra, probability, and calculus, both in the context of classroom investigations as well as in after-school sessions (for details, see Maher, 2005). Throughout the study, the mathematical concepts around which the tasks were designed preceded their introduction to the students in their regular school curriculum.

In the counting and combinatorics strand, the students worked on problems from second grade through the end of high school (12th grade). During this 11-year period, researchers engaged them in approximately 20 sessions in this strand of counting and combinatorics tasks. A set of problems in this strand that is relevant to this study, called Towers Problems, asks how many different towers of interlocking cubes (for example, Unifix® cubes[2]) can be built of a particular height when selecting from a certain number of colors of the cubes and to justify the solution obtained. From grades 3 to 10, the participants worked on versions of this problem with varied conditions. The first version given to the students was the four-tall towers problem, which they received when they were about nine years old (grade 3) in the following form:

> *Your group has two colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from two colors. See if*

you and your partner can plan a good way to find all the towers four cubes tall.

Then in grades four, five, and ten, they revisited this problem and worked on variations of it such as the five-tall towers problem with two colors, the four-tall towers problem with three colors, the *n*-tall towers problem with *k* colors, and this one:

Find all possible towers that are 5 cubes tall, selecting from two colors with exactly two of one color in each tower. Convince us that you have found them all.

For these Towers Problems, the students over time built knowledge of their underlying, similar mathematical structure.

Previously reported results

It is necessary to know how the students reason about the Towers Problem and how they connect it to Pascal's triangle to appreciate the mathematical ideas and reasoning evidenced in the conversation of the students presented in the results section of this article. In the fourth grade, they began to use a case-based reasoning and various counting strategies to justify that they had constructed all possible different towers five-tall when selecting from cubes of two different colors (red and yellow). The number of red cubes in the towers defined their cases. For instance, they found 32 different five-tall towers and, for the case of towers containing exactly two red and 3 yellow cubes, they reasoned that there are 10 different five-tall towers. Challenged to justify this result, the students reasoned again by cases, explaining that there are four different five-tall towers with two red cubes "stuck" together, three with one yellow cube between the two red cubes, two with two yellow cubes between the two red cubes, and one with three yellow cubes between the two red cubes. In this instance, the number of yellow cubes between the two red cubes defines each case.

The next time they visited the five-tall towers problem was in the tenth grade, two years before they worked on the problem that is the focus of this report. At that time, students were asked to revisit a question they answered in the fourth grade: how many towers could be built five-tall when selecting from cubes available in yellow and red that contained exactly two red cubes. Two students, one of whom is a participant in the present study, used 0s and 1s to represent respectively yellow and red cubes (see Figure 1) as well as use cases to organize their towers and justify their solution.

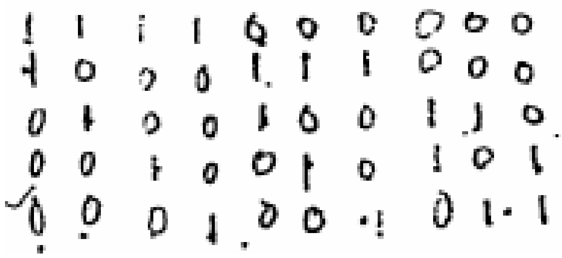


Figure 1 – Michael and Ankur's written work for all five-tall towers containing exactly two red and three yellow cubes.

Ankur and Michael discuss their work given in Figure 1. As Muter (1999) reports, they explain that a red cube is held fixed in the top position and then, starting in the next lower position, the second red cube is moved successively into the lower positions until it reaches the bottom position. Afterward, this process is repeated each time with the fixed red cube in the next lower position until the two red cubes occupy the last two positions at the bottom. Here, six years after they first developed reasoning by cases, their case-based argument concerns the fixed position of one of the one red cubes.

After the students explained their solution, a researcher introduced them to combinatorial notation. Uptegrove (2004) notes that in the previous session, the researcher introduced them to the expression

$$(a+b)^n$$

and that for $n = 2$ and 3 the students expanded the binomial expressions into polynomials. In this session, the researcher explained that the question of the number of different five-tall towers with exactly two red cubes is equivalent to asking how many combinations there are when selecting two object from a set of five different objects and presented how to represent this idea with combinatorial notation in four ways:

$${}_5C_2,\ C_{(5,2)},\ \binom{5}{2},\ C_2^5$$

She then showed the students how the binomial expansion and Pascal's triangle are related by the correspondence for $n = 0, 1, 2$, and 3, wrote out the numerical entries of the first five rows of Pascal's triangle, and invited the students to link their work on the Towers Problem to Pascal's triangle.

The students were able to make these links. They noticed the 10 that appears in the fifth row in Pascal's triangle corresponding to the expression

$$\binom{5}{2}$$

also presented the number of different five-tall towers with two red blocks (and three yellow cubes). They related the four-tall towers problem to the fourth row of Pascal's triangle – 1 4 6 4 1 – to subsets of 4-tall towers when choosing from two colors of Unifix® cubes (red and yellow). For instance, the students say that the 1 represents the one tower consisting of four red cubes, and the other 1 is for the tower consisting of four yellow cubes. This was the first time that the students connect a specific version of the Towers Problem to numerical entries in Pascal's triangle.

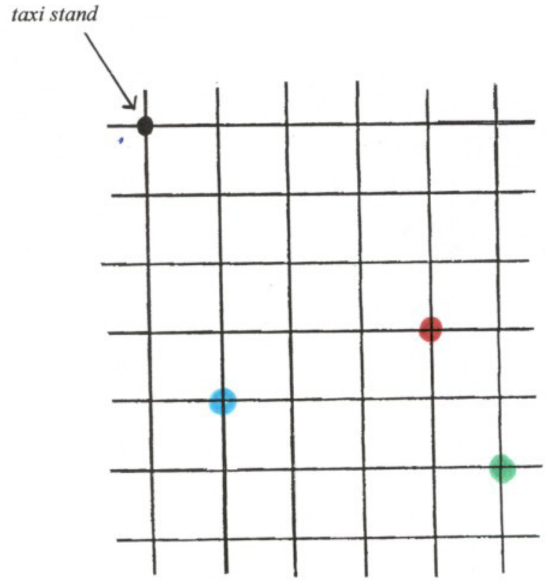
In summary, over the course of time, from fourth to tenth grade, the students developed a variety of mathematical ideas and ways of reasoning about counting and combinatorial problems. Among them, they invent counting strategies, discovered reasoning by cases, and investigated relationships between Towers Problems and Pascal's triangle.

Method

The research session of the present study occurs after school hours, at the end of the last year in high school. The students collaborate on the following culminating task of the research strand on combinatorics – The Taxicab Problem:[3]

> *A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.*

Accompanying the problem statement, the participants also have a map, actually, a 6 x 6 rectangular grid on which the left, uppermost intersection point represents the taxi stand (see below). The three passengers are positioned at different intersections, from left to right, as dots colored blue, red, and green, respectively, while their respective distances from the taxi stand are one unit east and four units south, four units east and three units south, and five units east and five units south.



In addition to the problem, the data sources consist of a video record, consisting of 1 hour and 16 minutes of the activity of the four participants from the perspective of two video cameras; a transcript of the videotapes combined to produce a fuller, more accurate verbatim record of the research session; the participants' inscriptions; and researcher field notes. The transcript is a textual rendering of verbal interactions, specifically, turn exchanges among the participants and between them and researchers, which in all consists of 1,619 turns at talk. It should be noted that for approximately 77% of the time the students spent in discursive interaction with each other and 92% of this time without the physical presence of the researchers.

The analytic method employs a sequence of phases, informed by grounded theory (Charmaz; Mitchell, 2001), ethnography and microanalysis (Erickson, 1992), an etic perspective based on an augmented framework of Gattegno's notion of the contents of mathematical experiences (Gattegno, 1987; Powell, 2003), and an approach for analyzing video data (for an elaboration and examples of the analytic phases, see Powell; Francisco; Maher, 2004).

Besides the non-Euclidean geometric setting, the Taxicab Problem has an underlying mathematical structure and encompasses concepts that resonate with those of other problems the participants have worked on in the longitudinal study (for details, see Maher, 2005). Their implicit task was to formulate and test conjectures. Researchers explicitly announced that they were to explain and justify conclusions. After they worked on the problem for about an hour and a half, researchers listened as they presented their resolution and asked questions to follow movements in their discourse toward further justification of their solution. Their resolution goes beyond the problem task: They generalize it and propose isomorphic propositions. It is in both of these actions that the students evidence of socially emergent cognition.

A final methodological issue concerns the role of the teacher-researcher. In the context of this investigation, student-to-student mathematical discussions involve a teacher-researcher who engages in minimal substantive interaction with the students while they collaborate on the given task. The teacher-researcher invites students to work on a mathematical task and is available to answer clarifying questions about the task. Later, the teacher-researcher listens to students describe their resolution of the task, how they arrived at it, and how the justify it.

Results

In an earlier analysis of these data from the analytic lens of the four categories of interlocution, Powell and Maher (2002) have illustrated that the conversational interactions among learners can advance their subsequent individual and collective actions. In particular, their analysis suggests that among the four interlocutory categories interpretive and negotiatory[4] interlocution

> *have the potential for advancing the mathematical understanding of individual learners working in a small group, the personal or individual understanding of a learner is intermeshed with the understanding of his or her interlocutors, and the mathematical ideas and understanding of an individual and his or her group emerge in a parallel fashion.* (p. 328, emphasis added)

These three consequences of interpretive and negotiatory interlocution evoke the construct, *socially emergent cognition.* As will be shown, socially emergent cognition is possible when interlocutors are engaged in negotiatory interlocution. This type of conversational exchange provides the discursive space for ideas and reasoning to emerge that go beyond those of any individual interlocutor and that are subsequently reflected in the discourse of individual interlocutor.

The four participants are Michael, Romina, Jeff, and Brian. In the first four minutes of the research session, the researcher spends little time at the table with the students and responds only to student questions in a tailored yet sparse manner. From then until 64 minutes later, the students engage with themselves. They rather quickly organize themselves, requesting colored markers and assigning subtasks to each other. Jeff inquires about why the routes to the blue destination point have the same length, Michael explains. Romina requests help in devising an area, Jeff and Michael respond and inform her that the applicable notion is perimeter, not that of area. In general, the students carefully and respectfully listen and respond to each other's questions, statements, and ideas.

In the session, the first example of socially emergent cognition occurs after 14 minutes into the problem solving session. After almost 14 minutes into the research session, there is an interesting and pivotal interaction among Romina, Brian, and Jeff:

Episode 1[5]

Romina: I think we're going to have to break it apart and draw as many as possible.

Brian: Yeah, //that's what I'm going to do.

Jeff: // And then have that lead us to something? What if we do- why don't we do easier ones? You know what I'm saying? What if the- the thing- Do you have another one of these papers?

In Episode 1, an agenda for action emerges from the students' interlocution. Brian and Jeff accept the task implied in Romina's statement and act on her heuristic. Furthermore, Jeff refines her suggestion in his interrogative: "why don't we do easier ones?" Romina's statement and Jeff's interrogative establish a new agenda for the group's actions. Importantly, this action agenda represents a watershed in their mathematical investigation. From this point onward, they no longer work on the combinatorial problem as given but instead pose and work on simpler situations to glean relevant information and extract insights from those situations so as to inform their understanding and resolution of the given problem. This agenda emerges from the students' negotiatory interlocution. It was not posed fully formed by any one student. However, after its materialization from the socially emergent cognition of the group forms part of the students' understanding of how they will proceed to resolve the given problem.

Another instance of socially emergent cognition transpires over many turns of speech, spanning from about turn 159 to turn 1320. Space does not permit a full illustration of the development of the ideas and reasoning that comprise the students' socially emergent cognition. They have continual discursive interactions with the aim of building an isomorphism between a rule for generating the entries of Pascal's triangle and the number of shortest routes to points on the taxicab grid. Early in their work, the students manifest embryonic thinking about an isomorphism. Romina wonders aloud: "can't we do towers[6] on this" (turn 159).[7] Her public query catalyzes a negotiatory interlocution among Michael, Jeff, and her. Jeff, responding immediately to Romina, says, "that's what I'm saying," (turn 160) and invites her to think with him about the dyadic choice ("there or there" turn 162) that one has at intersections of the taxicab grid. Furthermore, he wonders whether one can find the number of shortest routes to a pick-up point by adding up the different choices one encounters in route to the point (turn 162). Romina proposes that since the length of a shortest route to the red pick-up point is 10, then "ten could be like the number of blocks we have in the tower" (turn 169). Romina's query concerning the application of towers to the present problem task prompts Michael's engagement with the idea, as well. As if advising his colleagues and himself, he reacts in part by saying, "think of the possibilities of doing this and then doing that" (turn 180). While uttering these words, he points at an intersection; from that intersection gestures first downward ("doing this"), returns the to point, and then motions rightward ("doing that"). Similar to Jeff's words and gestures, Michael's actions also acknowledge cognitively and corporally the dyadic-choice aspect of the problem task. Through their negotiatory interactions, Michael, Jeff, and Romina raised the prospect of as well as provided insights for building an isomorphism between the Taxicab and Towers Problems.

The prospect and work of building such an isomorphism reemerges several more times in the participants' interlocution, and each time, they further elaborate their insights and advance more isomorphic propositions. Eventually, the building of isomorphisms dominates their conversational exchanges. Approximately thirty-five minutes after Romina first broached the possibility of relating attributes of the Towers Problem to the problem at hand, the participants reengage with the idea. Romina speculates that between the two problems one can relate "like lines over" to "like the color" and then "the lines down" to the "number of blocks"(turn 738). What is essential here is Romina's apparent awareness that each of the two different directions of travel in the Taxicab Problem needs to be associated with different objects in the Towers Problem.

Romina uses this insight later in the session. She transfers the data that she and her colleagues have generated from a transparency of a 1-centimeter grid to plain paper. Their data are equivalent to binomial coefficients. She identifies one unit of horizontal distance with one Unifix® cube of color *A* and one unit of vertical distance with one Unifix® cube of color *B*:

> *Like doesn't the two- there's- that I mean, that's one- that means it's one of A color, one of B color [pointing to the 2 in Pascal's triangle]. Here's one- it's either one- either way you go. It's one of across and one down [pointing to a number on the transparency grid and motions with her pen to go across and down]. And for three that means there's two A color and one B color [pointing to a 3 in Pascal's triangle], so here it's two across, one down or the other way [tracing across and down on the transparency grid] you can get three is two down [pointing to the grid].* (turn 1210)

Furthering the building of their isomorphism, Michael offers another propositional foundation. Pointing at their data on the transparency grid and referring to its diagonals as rows, he notes that each row of the data refers to the number of shortest routes to particular points of a particular length. For instance, pointing the array – 1 4 6 4 1 – of their transparency, he observes that each number refers to an intersection point whose "shortest route is four" (turn 1203). Moreover, he remarks that one could name a diagonal by, for example, "six" since "everything [each intersection point] in the row [diagonal] has shortest route of six"(turn 1205). In terms of an isomorphism, Michael's observation points in two different directions: (1) it relates diagonals of information in their data to rows of numbers in Pascal's triangle and (2) it notes that intersection points whose shortest routes have the same length can have different numbers of shortest routes.

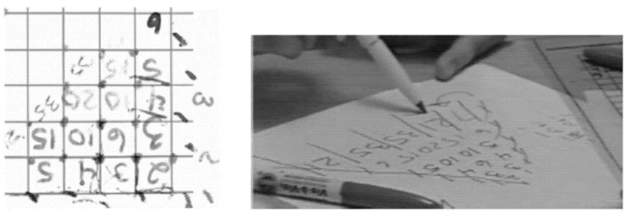


Figure 2 – Participant's data arrays (from their perspective): (A) In green, empirical data of shortest routes between the taxi stand and nearby intersection points. Jeff wrote the ones in blue to augment the appearance of the numerical array as Pascal's triangle. From the participant perspective, to the left of Jeff's numbers, Romina wrote in green the numbers 1, 2, and 3 to indicate the row numbers of the triangular array. (B) The first five rows contain empirical data;

the remaining two rows contain assumed data values based on the addition rule for Pascal's triangle.

Later in responding to a researcher's question, the participants develop a proposition that relates how they know that a particular intersection in the taxicab grid corresponds to a number in Pascal's triangle. They focus their attention on their inscriptions, A and B, in Figure 2. Michael and Romina discuss correspondences between the two inscriptions. Referring to a point on their grid that is five units east and two units south, Romina associates the length of its shortest route, which is seven, to a row of her Pascal's triangle by counting down seven rows and saying, "five of one thing and two of another thing" (turn 1313). Michael inquires about her meaning for "five and two" (turn 1314). Both Romina and Brian respond, "five across and two down" (turns 1317 and 1318). She then associates the combinatorial numbers in the seventh row of her Pascal's triangle to the idea of "five of one thing and two of another thing," specifying that, left to right from her perspective, the first 21 represents two of one color, while the second 21 "is five of one color" (turn 1320), presuming the same color. Using this special case, Romina hints at a general proposition for an isomorphism between the Taxicab and Towers Problems.

Discussion

The above presents evidence that through negotiatory interlocution students build an isomorphism during the course of the problem-solving session. The content of the phases include the following with indication of when from the start of the session each occurs: (1) there exists a relationship between the Towers and Taxicab Problems, (turn 159); (2) Similar to the Towers Problem, the Taxicab Problem has a dyadic choice, (turns 162 and 180); (3) The length of a shortest route to an intersection point corresponds to the height of a tower, (turn 169) [0:08:15]; (4) Each of the two different directions of travel in the Taxicab Problem needs to be associated with different objects in the Towers Problem, (turn 738); (5) Rebuild the meaning of 2 to the *n* in the environment of the Towers Problem, (turns 171 and 742-748); (6) Identify one unit of horizontal distance with one Unifix\ cube of color *A* and one unit of vertical distance with one Unifix\ cube of color *B*, (turn 1210); (7) A row "diagonal" of their data contains the number of shortest routes for intersection points whose shortest distance from the taxi stand is *n*, (turn 1203); (8) Intersection points whose shortest routes have the same length can have different numbers of shortest routes, (turn 1205); (9) A tower 3-high with 2 of one color and 1 of another color corresponds to routes to a point 2 down and 1 across, (turn 1214); and (10) Intersection point five units east and two south from the taxi stand corresponds to five of one thing and two of another thing and, therefore, go the seventh row of Pascal's triangle and the second and fifth entries of the triangle to find the number of shortest routes from the taxi stand to the intersection point five units east and two south from the taxi stand, (turns 1309-1320).

The isomorphism that the students build is constituted from their negotiatory discursive interaction and represents and example of the theoretical construct – socially emergent cognition. Not one student presents the isomorphism fully formed, but rather their discursive interactions constitute a co-construction of the isomorphism. It can be observed that early in the problem-solving session the three participants – Romina, Jeff, and Michael – articulate awareness of object and relational connections between their current problem task and a former one, the Towers Problem. Later, upon noticing that their array of data resembles Pascal's triangle and conjecturing so, the participants embark on building an isomorphism between the Towers Problem and the Taxicab Problem as an approach to justifying their conjecture since from previous experience they know that Pascal's triangle underlies the mathematical structure of the Towers Problem. In this sense, their strategy can be interpreted as justifying their conjecture by transitivity: (a) Pascal's triangle is equivalent to Towers and (b) Towers is equivalent to Taxicab; therefore implying that (c) Pascal's triangle is equivalent to Taxicab. They know (a) is true and embark on demonstrating (b) to justify and conclude (c).

Notas

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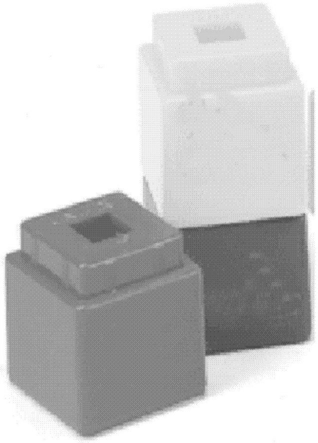
[2] Unifix® cubes are manipulative materials of plastic cubes that interlock along one axis. They are commercially available <didax.com/unifix/>, come an assortment of colors, and look like this:



[3] This problem task is similar to ones that Kaleff and Nascimento (2004) discuss.
[4] In Powell and Maher (2002), the category of "negotiatory" interlocution is called "hermeneutic."
[5] The transcription symbol "//," a double slash mark, indicates the moment at which overlapping speech begins, and a dash "-" signals an interruption in a speaker's utterance.
[6] Here Romina refers to a genre of problems, called the Towers Problems, discussed earlier in the Background section.
[7] For Romina and other participants in the longitudinal study, this comment is pregnant with mathematical and heuristic meaning derived from their constructed, shared experiences with tasks and inscriptions in the combinatorial and probability strands of the longitudinal study (see, for instance, Kiczek, 2000; Martino, 1992; Muter, 1999 and Uptegrove, 2004).

References

ALRØ, H.; SKOVSMOSE, O. That was not the intention! Communication in mathematics education. *For the Learning of Mathematics,* v. 18, n. 2, p. 41-51, 1998.

CHARMAZ, K.; MITCHELl, R. G. Grounded theory in ethnography. In: ATKINSON, P.; COFFEY, A.; DELAMONT, S. (Ed.). *Handbook of ethnography*. London: Sage, 2001. p. 160-174.

COBB, P.; BOUFI, A.; MCCLAIN, K.; WHITENACK, J. Reflective discourse and collective reflection. *Journal for Research in Mathematics Education,* v. 28, n. 3, p. 258-277, 1997.

DAMON, W.; PHELPS, E. Peer collaboration as a context for cognitive growth. In: LANDSMANN, L. T. (Ed.). *Culture, schooling, and psychological development.* Norwood, NJ: Ablex, 1991. p 171-184.

DAVIS, B. Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education,* v. 28, n. 3, p. 355-376, 1997.

ERICKSON, F. The Interface between ethnography and microanalysis. In: LECOMPTE, M. D.; MILLROY, W. L.; PREISSLE, J. (Ed.). *The handbook of qualitative research in education.* San Diego: Academic Press, 1992. p. 201-225.

FERNANDEZ, E. A kinder, gentler Socrates: Conveying new images of mathematics dialogue. *For the Learning of Mathematics,* v. 14, n. 3, p. 43-46, 1994.

GATTEGNO, C. *The science of education:* Part 1: Theoretical considerations. New York: Educational Solutions, 1987.

HUTCHINS, E. *Cognition in the wild.* Cambridge: MIT, 1995.

KALEFF, A. M.; NASCIMENTO, R. S. Atividades introdutórias às geometrias não-euclidianas: o exemplo da geometria do táxi. *Boletim Gepem,* v. 44, p. 11-42, 2004.

KICZEK, R. D. *Tracing the development of probabilistic thinking:* Profiles from a longitudinal study. 272 f. Tese (Doutorado em Educação Matemática)– Rutgers, the State University of New Jersey, New Brunswick, 2000.

MAHER, C. A. Kommunikation och konstruktivistisk undervisning [Communication and constructivist teaching]. In: ENGRSTOM, A. (Ed.). *Matematik och reflektion:* En introduktion till konstruktivismen inom matematikdidaktiken. Lund, Sweden: Studenlitteratur, 1998, p. 124-143.

MAHER, C. A. How students structure their investigations and learn mathematics: Insights from a long-term study. *The Journal of Mathematical Behavior,* v. 24, n. 1, p. 1-14, 2005.

MAHER, C. A.; Martino, A. M. From patterns to theories: Conditions for conceptual change. *Journal of Mathematical Behavior*, v. 19, n. 2, p. 247-271, 2000.

MARTINO, A. M. *Elementary students' construction of mathematical knowledge:* Analysis by profile. 358 f. Tese (Doutorado em Educação Matemática)– Rutgers, the State University of New Jersey, New Brunswick, 1992.

MUTER, E. M. *The development of student ideas in combinatorics and proof:* A six-year study. 210 f. Tese (Doutorado em Educação Matemática)– Rutgers, the State University of New Jersey, New Brunswick, 1999.

O'CONNOR, M. C.; MICHAELS, S. Shifting participant frameworks: Orchestrating thinking practices in group discussion. In: HICKS, D. (Ed.). *Discourse, learning, and schooling.* New York: Cambridge University, 1996. p. 63-103.

PIRIE, S.; SCHWARZENBERGER, R. Mathematical discussion and mathematical understanding. *Educational Studies in Mathematics,* v. 19, n. 4, p. 459-470, 1998.

POWELL, A. B. *"So let's prove it!":* Emergent and elaborated mathematical ideas and reasoning in the discourse and inscriptions of learners engaged in a combinatorial task. 361 f. Tese (Doutorado em Educação Matemática)– The State University of New Jersey, New Brunswick, 2003.

POWELL, A. B.; MAHER, C. A. Inquiry into the interlocution of students engaged with mathematics: Appreciating links between research and practice. In: MEWBORN, D. S.; SZTAJN, P.; WHITE, D. Y.; WIEGEL, H. G.; BRYANT, R. L.; NOONEY, K. (Ed.). *Proceedings of the twenty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Athens, Georgia).* Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, 2002. v. 1, p. 317-329.

POWELL, A. B.; FRANCISCO, J. M.; MAHER, C. A. Uma abordagem à análise de dados de vídeo para investigar o desenvolvimento das idéias matemáticas e do raciocínio de estudantes [An analytical model for studying the development of mathematical ideas and reasoning using videotape data]. *Bolema: O Boletim de Educação Matemática [Bolema: The Bulletin of Mathematics Education],* v. 21, p. 81-140, 2004.

SEEGER, F. Research on discourse in the mathematics classroom: A commentary. *Educational Studies in Mathematics,* v. 49, n. 1-3, p. 287-297, 2002.

SFARD, A. On reform movement and the limits of mathematical discourse. *Mathematical Thinking and Learning,* v. 2, n. 3, p. 157-189, 2000.

SFARD, A. There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics,* v. 46, p. 13-57, 2001.

SFARD, A.; NESHER, P.; STREEFLAND, L.; COBB, P.; MASON, J. Learning mathematics through conversation: Is it as good as they say? *For the Learning of Mathematics,* v. 18, n. 1, p. 41-51, 1998.

STAHL, G. Group cognition in computer-assisted collaborative learning. *Journal of Computer Assisted Learning,* v. 21, p. 79-90, 2005.

UPTEGROVE, E. B. *To symbols from meaning:* Students' long-term investigations in counting. 532 f. Tese (Doutorado em Educação Matemática)– Rutgers, The State University of New Jersey, New Brunswick, 2004.

VYGOTSKY, L. S. *Mind in society:* The development of higher psychological processes. Trad. M. Cole, V. John-Steiner e S. Scribner, E. Cambridge, MA: Harvard, 1978.

WEBER, K; MAHER, C. A.; POWELL, A. B. Strands of challenging mathematical problems and the construction of mathematical problem-solving schema. Trondheim, Norway: ICMI 16 Study. No prelo.

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